

A NOTE ON CUSP FORMS AS p -ADIC LIMITS

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ABSTRACT. Several authors have recently proved results which express cusp forms as p -adic limits of weakly holomorphic modular forms under repeated application of Atkin's U -operator. The proofs involve techniques from the theory of weak harmonic Maass forms, and in particular a result of Guerzhoy, Kent, and Ono on the p -adic coupling of mock modular forms and their shadows. Here we obtain strengthened versions of these results using techniques from the theory of holomorphic modular forms.

1. INTRODUCTION

In a recent paper [4], El-Guindy and Ono study a cusp form and a modular function related to the elliptic curve $y^2 = x^3 - x$. Following their notation, define

$$g(z) = \eta^2(4z)\eta^2(8z) = \sum_{n \geq 1} a(n)q^n = q - 2q^5 - 3q^9 + 6q^{13} + \cdots, \quad (1.1)$$

$$L(z) = \frac{\eta^6(8z)}{\eta^2(4z)\eta^4(16z)} = \frac{1}{q} + 2q^3 - q^7 - 2q^{11} + \cdots, \quad (1.2)$$

$$F(z) = -g(z)L(2z) = \sum_{n \geq -1} C(n)q^n = -\frac{1}{q} + 2q^3 + q^7 - 2q^{11} + \cdots. \quad (1.3)$$

The main result of [4] states that if $p \equiv 3 \pmod{4}$ is a prime for which $p \nmid C(p)$, then as a p -adic limit, we have

$$\lim_{m \rightarrow \infty} \frac{F|U(p^{2m+1})}{C(p^{2m+1})} = g. \quad (1.4)$$

The proof involves the theory of harmonic Maass forms, and in particular a result of Guerzhoy, Kent, and Ono [5] on the p -adic coupling of mock modular forms and their shadows. Similar results were proved in [5] and [2].

Our goal is to prove strengthened versions of these results. We use a direct method; it does not involve harmonic Maass forms but rather an investigation of the action of the Hecke operators on a family of weakly holomorphic modular forms. A similar approach was recently employed in the study of the congruences of Honda and Kaneko [1]. For the modular forms described above, we prove the following, of which (1.4) is an immediate corollary. Note in addition that the $m = 0$ case of (1.5) gives $p \nmid C(p)$. Let $v_p(\cdot)$ denote the p -adic valuation on $\mathbb{Z}[[q]]$.

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Theorem 1.1. *Let $p \equiv 3 \pmod{4}$ be prime. Then for all integers $m \geq 0$ we have*

$$v_p(C(p^{2m+1})) = m, \quad (1.5)$$

$$v_p\left(\frac{F|U(p^{2m+1})}{C(p^{2m+1})} - g\right) \geq m + 1. \quad (1.6)$$

In Theorems 4.1 and 5.1 below we obtain similar improvements of results given in [5] and [2]. It is clear that the present approach would give similar results for a number of other spaces of modular forms.

2. BACKGROUND

If k is an integer, f is a function of the upper half-plane, and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$, we define

$$f(z)|_k \gamma := (\det \gamma)^{k/2} (cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right).$$

If $N \geq 1$, $k \in \mathbb{Z}$, and χ is a Dirichlet character modulo N , let $M_k(N, \chi)$ be the space consisting of functions f which satisfy $f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi(d)f$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and which are holomorphic on the upper half plane and at the cusps. Let $M_k^!(N, \chi)$ be the space of forms which are meromorphic at the cusps, and let $M_k^\infty(N, \chi)$ denote the subspace of forms which are holomorphic at all cusps of $\Gamma_0(N)$ other than ∞ . We drop the character from this notation when it is trivial. Each $f \in M_k^!(N, \chi)$ can be identified with its q -expansion; with $q := \exp(2\pi iz)$ we have $f(z) = \sum a(n)q^n$ for some coefficients $a(n)$.

For each positive integer m , the U and V -operators are defined on q -expansions by

$$\begin{aligned} \sum a(n)q^n |U(m) &:= \sum a(mn)q^n, \\ \sum a(n)q^n |V(m) &:= \sum a(n)q^{mn}. \end{aligned}$$

Let $T_{k,\chi}(m)$ be the usual Hecke operator on $M_k^!(N, \chi)$. If p is prime, then for $n \geq 1$ and $f \in M_k^!(N, \chi)$ we have

$$f|T_{k,\chi}(p^n) = \sum_{j=0}^n \chi(p^j) p^{(k-1)j} f|U(p^{n-j})|V(p^j). \quad (2.1)$$

Define

$$\Theta := \frac{1}{2\pi i} \frac{d}{dz} = q \frac{d}{dq}.$$

Lemma 2.1. *If $(m, N) = 1$, then we have*

$$T_{k,\chi}(m) : M_k^\infty(N, \chi) \rightarrow M_k^\infty(N, \chi). \quad (2.2)$$

If $k \geq 2$ then

$$\Theta^{k-1} : M_{2-k}^\infty(N, \chi) \rightarrow M_k^\infty(N, \chi). \quad (2.3)$$

Proof. For the first statement, it suffices to show that for each prime $p \nmid N$ we have

$$T_{k,\chi}(p) : M_k^\infty(N, \chi) \rightarrow M_k^\infty(N, \chi).$$

We have

$$f|T_{k,\chi}(p) = p^{\frac{k}{2}-1} \left(\sum_{j=0}^{p-1} f|_k \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} + \chi(p) f|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right). \quad (2.4)$$

Let $r \in \mathbb{Q}$ be a cusp of $\Gamma_0(N)$ inequivalent to ∞ and choose $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \setminus \Gamma_0(N)$ with $\gamma\infty = r$. Given $j \in \{0, \dots, p-1\}$ set $\lambda := (a + cj, p)$. By a standard argument (see e.g. [6, §6.2]) we find that

$$\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{a+cj}{\lambda} & * \\ \frac{cp}{\lambda} & * \end{pmatrix} \begin{pmatrix} \lambda & * \\ 0 & \frac{p}{\lambda} \end{pmatrix}$$

where the first matrix on the right is in $\mathrm{SL}_2(\mathbb{Z}) \setminus \Gamma_0(N)$. It follows that each term from the sum on j in (2.4) is holomorphic at cusps other than ∞ . To see that the last summand is also holomorphic at these cusps, let $\lambda' := (p, c)$. Then

$$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{ap}{\lambda'} & * \\ \frac{c}{\lambda'} & * \end{pmatrix} \begin{pmatrix} \lambda' & * \\ 0 & \frac{p}{\lambda'} \end{pmatrix}$$

where the first matrix on the right is in $\mathrm{SL}_2(\mathbb{Z}) \setminus \Gamma_0(N)$.

Let R_k be the Maass raising operator in weight k , so that we have the basic relation

$$R_{k-2}(f|_{k-2}\gamma) = (R_{k-2}f)|_k\gamma.$$

Bol's identity (see for example [3, Lemma 2.1]) states that for $k \geq 2$ we have

$$\Theta^{k-1} = \frac{1}{(-4\pi)^{k-1}} R_{k-2} \circ R_{k-4} \circ \dots \circ R_{4-k} \circ R_{2-k}.$$

It follows that

$$\Theta^{k-1} : M_{2-k}^!(N, \chi) \rightarrow M_k^!(N, \chi)$$

and that

$$(\Theta^{k-1}f)|_k\gamma = \Theta^{k-1}(f|_{2-k}\gamma).$$

The claim (2.3) follows from these two facts. \square

If $p \nmid 6N$ and $k \geq 0$, let $M_k^{(p)}(N)$ denote the subset of $M_k(N)$ consisting of forms whose coefficients are p -integral rational numbers. If $f \in M_k^{(p)}(N)$, define the filtration

$$w_p(f) := \inf\{k' : f \equiv g \pmod{p} \text{ for some } g \in M_{k'}^{(p)}(N)\}.$$

We require two facts, which can be found for example in [7, §1]. First, if $f \in M_k^{(p)}(N)$ and $w_p(f) \neq -\infty$, then $w_p(f) \equiv k \pmod{p-1}$. Also, we have

$$w_p(f|V(p)) = p w_p(f). \quad (2.5)$$

3. PROOF OF THEOREM 1.1

Recall the definitions (1.1)–(1.3), and note that $F = F_1$ and $g = -F_{-1}$ in the notation of the next proposition.

Proposition 3.1. *We have the following.*

(1) *For every odd integer $m \geq -1$ there exists a unique $F_m \in M_2^\infty(32) \cap \mathbb{Z}[[q]]$ of the form*

$$F_m = -q^{-m} + O(q^3).$$

(2) *Suppose that p is an odd prime and that $n \geq 0$. Then*

$$F|T_2(p^n) = p^n F_{p^n} + C(p^n)g.$$

Proof. For each integer $r \geq 0$, let

$$E_r(z) = -g(z)L^r(2z) = -\frac{\eta^2(4z)\eta^{6r}(16z)}{\eta^{2r-2}(8z)\eta^{4r}(32z)} = -q^{-2r+1} + 2q^{-2r+5} + O(q^{-2r+9}).$$

Using standard criteria (see, e.g. [8, Thm. 1.64, Thm. 1.65]) we find that $E_r \in M_2^\infty(32)$. The forms F_m can then be constructed as linear combinations of forms E_r with $2r - 1 \equiv m \pmod{4}$. Uniqueness follows since the space $S_2(32)$ is one-dimensional. This gives the first assertion.

From (2.1) we have

$$F|T_2(p^n) = F|U(p^n) + \sum_{j=1}^{n-1} p^j F|U(p^{n-j})|V(p^j) + p^n F|V(p^n).$$

Observe that

$$F|U(p^n) = C(p^n)q + O(q^3) = C(p^n)g + O(q^3)$$

and that

$$\sum_{j=1}^{n-1} p^j F|U(p^{n-j})|V(p^j) + p^n F|V(p^n) = -p^n q^{-p^n} + O(q^3).$$

Assertion (2) follows from assertion (1) together with Lemma 2.1. \square

Before proving Theorem 1.1 we require two lemmas.

Lemma 3.2. *For each prime $p \equiv 3 \pmod{4}$ and each integer $m \geq 0$ we have*

$$C(p^{2m+1}) \equiv (-1)^m p^m C(p) \pmod{p^{m+1}}.$$

Proof. Lemma 2.3 and Corollary 2.4 of [4] show that for each $p \equiv 3 \pmod{4}$, there is a modular function $\phi_p \in M_0^\infty(32)$ of the form

$$\phi_p(z) = q^{-p} + C(p)q + O(q^3) \quad (3.1)$$

(we have corrected a sign error in the proof of the corollary). From Lemma 2.1 we have

$$\Theta(\phi_p) = -pq^{-p} + C(p)q + O(q^3) \in M_2^\infty(32).$$

On the other hand, Proposition 3.1 gives

$$F|T_2(p) = -pq^{-p} + C(p)q + O(q^3).$$

Therefore

$$F|T_2(p) = \Theta(\phi_p), \quad (3.2)$$

or equivalently

$$F|U(p) = \Theta(\phi_p) - p F|V(p). \quad (3.3)$$

Applying $U(p^2)$ to both sides of (3.3) and arguing inductively, we obtain the following for each $m \geq 0$:

$$F|U(p^{2m+1}) = \sum_{k=0}^m (-1)^{m-k} p^{m-k} \Theta(\phi_p)|U(p^{2k}) + (-1)^{m+1} p^{m+1} F|V(p). \quad (3.4)$$

For any $k \geq 0$ we have $\Theta(\phi_p)|U(p^{2k}) \equiv 0 \pmod{p^{2k}}$. Therefore for each $m \geq 0$ we have

$$F|U(p^{2m+1}) \equiv (-1)^m p^m \Theta(\phi_p) \pmod{p^{m+1}}. \quad (3.5)$$

The lemma follows by comparing coefficients of q in (3.5). \square

The authors of [4] speculated that $v_p(C(p)) = 0$ for every prime $p \equiv 3 \pmod{4}$. We prove that this is the case.

Lemma 3.3. *For each prime $p \equiv 3 \pmod{4}$ we have $p \nmid C(p)$.*

Proof. Assume to the contrary that $p \mid C(p)$. From (3.2) and Proposition 3.1 it follows that

$$\Theta(\phi_p) = F|T_2(p) = pF_p + C(p)g \equiv 0 \pmod{p},$$

from which it follows that for some integral coefficients A_p we have

$$\phi_p \equiv q^{-p} + \sum_{n=1}^{\infty} A_p(np)q^{np} \pmod{p}.$$

Let

$$f(z) = \frac{\eta^8(32z)}{\eta^4(16z)} = q^8 + 4q^{24} + O(q^{40}) \in M_2(32).$$

Then $f^p \in M_{2p}(32)$ has the form

$$f^p \equiv \sum_{n=8}^{\infty} B_p(np)q^{np} \equiv q^{8p} + \dots \pmod{p}.$$

Since $\phi_p \in M_0^\infty(32)$, we find that $h_p := \phi_p f^p \in M_{2p}(32)$ has the form

$$h_p \equiv \sum_{n=7}^{\infty} D_p(pn)q^{pn} \equiv q^{7p} + \dots \pmod{p}.$$

so that

$$h_p \equiv h_p|U(p)|V(p) \pmod{p}. \quad (3.6)$$

Using (2.5) we obtain

$$w_p(h_p) = p w_p(h_p|U(p)).$$

Since $w_p(h_p) \equiv 2p \pmod{p-1}$ and $p \mid w_p(h_p)$ we must have $w_p(h_p) = 2p$, so that $w_p(h_p|U(p)) = 2$. Thus there exists $h_0 \in M_2^{(p)}(32)$ such that

$$h_0 \equiv h_p|U(p) = q^7 + O(q^8) \pmod{p}.$$

However, by examining a basis for the eight-dimensional space $M_2(32)$ we find that there is no such form h_0 . This provides the desired contradiction. \square

Proof of Theorem 1.1. Assertion (1.5) follows from Lemmas 3.2 and 3.3. To prove (1.6), we use Proposition 3.1 and (2.1) to find that

$$\frac{F|U(p^{2m+1})}{C(p^{2m+1})} - g = \frac{1}{C(p^{2m+1})} \left(p^{2m+1} F_{p^{2m+1}} - \sum_{j=1}^{2m+1} p^j F|U(p^{2m+1-j})|V(p^j) \right). \quad (3.7)$$

Using (2.1) we obtain

$$F|T_2(p^{2m}) = \sum_{j=1}^{2m+1} p^{j-1} F|U(p^{2m+1-j})|V(p^{j-1}).$$

Since $C(n) = 0$ for $n \not\equiv 3 \pmod{4}$, we see from Proposition 3.1 that $F|T_2(p^{2m}) = p^{2m}F_{p^{2m}}$. It follows that

$$\sum_{j=1}^{2m+1} p^j F|U(p^{2m+1-j})|V(p^j) = p^{2m+1}F_{p^{2m}}|V(p) \equiv 0 \pmod{p^{2m+1}}.$$

Assertion (1.6) now follows from (3.7) and (1.5). \square

4. AN EXAMPLE IN WEIGHT 4 AND LEVEL 9

In [5], the authors study the p -adic coupling of mock modular forms and their shadows. As an application of their general result, they prove two p -adic limit formulas involving the hypergeometric functions ${}_2F_1(\frac{1}{3}, \frac{1}{3}; 1; z)$ and ${}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; z)$ evaluated at certain modular functions. We will use the following notation:

$$\begin{aligned} g_1(z) &= \eta^8(3z) = \sum_{n \geq 1} a(n)q^n = q - 8q^4 + 20q^7 - 70q^{13} + \cdots \in S_4(9), \\ L_1(z) &= \frac{\eta^3(z)}{\eta^3(9z)} + 3 = \frac{1}{q} + 5q^2 - 7q^5 + 3q^8 + 15q^{11} + \cdots, \\ G(z) &= g_1(z)L_1^2(z) = \sum_{n \geq -1} C(n)q^n = \frac{1}{q} + 2q^2 - 49q^5 + 48q^8 + 771q^{11} + \cdots. \end{aligned}$$

After rewriting using (3.3) and (3.4) of [5], we find that each of the two formulas in Theorem 1.3 of [5] is equivalent to the assertion that for every prime $p \equiv 2 \pmod{3}$ with $p^3 \nmid C(p)$ we have

$$\lim_{m \rightarrow \infty} \frac{G|U(p^{2m+1})}{C(p^{2m+1})} = g_1(z). \quad (4.1)$$

Here we prove a strengthened version of this result.

Theorem 4.1. *Let $p \equiv 2 \pmod{3}$ be a prime. Then for each integer $m \geq 0$ we have*

$$v_p(C(p^{2m+1})) = \begin{cases} 3m + 1, & \text{if } p = 2, \\ 3m, & \text{if } p \neq 2. \end{cases} \quad (4.2)$$

$$v_p\left(\frac{G|U(p^{2m+1})}{C(p^{2m+1})} - g_1\right) \geq \begin{cases} 3m + 2, & \text{if } p = 2, \\ 3m + 3, & \text{if } p \neq 2. \end{cases} \quad (4.3)$$

The proof follows the argument in Section 3, so we give fewer details here.

Proposition 4.2. *We have the following.*

- (1) *For every integer $m \geq -1$ with $3 \nmid m$, there exists a unique $G_m \in M_4^\infty(9) \cap \mathbb{Z}[[q]]$ of the form*

$$G_m = q^{-m} + O(q^2).$$

- (2) *Let $p \neq 3$ be prime and let n be a nonnegative integer. Then we have*

$$G|T_4(p^n) = p^{3n}G_{p^n} + C(p^n)g_1.$$

Proof. For each integer $r \geq 0$, let

$$E_r(z) = g_1(z)L_1(z)^r = q^{1-r} + (5r - 8)q^{4-r} + O(q^{7-r}).$$

Then $E_r(z) \in M_4^\infty(9)$. We construct each form G_m by taking a linear combination of E_r with $r - 1 \equiv m \pmod{3}$. Uniqueness follows since $S_4(9)$ is spanned by the form $g_1 = G_{-1}$.

We deduce assertion (2) as in the last section using (2.2), (2.1), and assertion (1). \square

Lemma 4.3. *If $p \equiv 2 \pmod{3}$ is prime, then*

$$C(p^{2m+1}) \equiv (-1)^m p^{3m} C(p) \pmod{p^{3m+3}}.$$

Proof. Define

$$\phi_2(z) = \frac{\eta^2(3z)}{\eta^6(9z)} = \sum_{n \geq -2} A_2(n) q^n = \frac{1}{q^2} - 2q - q^4 + O(q^5).$$

It is seen from the expression of ϕ_2 as an infinite product that $A_2(n) = 0$ if $n \not\equiv 1 \pmod{3}$. Similarly, if

$$L_1(z) = \frac{\eta^3(z)}{\eta^3(9z)} + 3 = \sum_{n \geq -1} b(n) q^n,$$

then $b(n) = 0$ for all $n \not\equiv 2 \pmod{3}$. Therefore, for each positive integer $l \equiv 2 \pmod{3}$ there exist $c_0, c_1, \dots, c_{\frac{l-2}{3}} \in \mathbb{Z}$ such that

$$\phi_l(z) = \phi_2(z) \sum_{j=0}^{\frac{l-2}{3}} c_j L_1^{l-2-3j}(z) = q^{-l} + \sum_{n \geq 1} A_l(n) q^n \in M_{-2}^\infty(9),$$

with $A_l(n) \in \mathbb{Z}$ and $A_l(n) = 0$ if $n \not\equiv 1 \pmod{3}$ (these coincide with the forms w_l in [5, Prop. 3.1]). Since the constant term in the weight two modular form $\phi_l L_1$ must be zero, we find as in the last section that $A_l(1) = -C(l)$. In particular, for any prime $p \equiv 2 \pmod{3}$ we have

$$\phi_p = q^{-p} - C(p)q + O(q^2).$$

By Lemma 2.1, we have

$$\Theta^3(\phi_p) = -p^3 q^{-p} - C(p)q + O(q^2) \in M_4^\infty(9).$$

Hence it follows from Proposition 4.2 that

$$\Theta^3(\phi_p) = -p^3 G_p - C(p)g_1 = -G|T_4(p) = -G|U(p) - p^3 G|V(p), \quad (4.4)$$

so that

$$G|U(p) = -\Theta^3(\phi_p) - p^3 G|V(p). \quad (4.5)$$

Applying $U(p^2)$ iteratively leads to

$$G|U(p^{2m+1}) = \sum_{l=0}^m (-1)^{m+1-l} p^{3(m-l)} \Theta^3(\phi_p)|U(p^{2l}) + (-1)^{m+1} p^{3(m+1)} G|V(p) \quad (4.6)$$

for any non-negative integer m . Since $\Theta^3(\phi_p)|U(p^{2l}) \equiv 0 \pmod{p^{6l}}$, we have from (4.6) that

$$G|U(p^{2m+1}) \equiv (-1)^{m+1} p^{3m} \Theta^3(\phi_p) \pmod{p^{3m+3}}. \quad (4.7)$$

Comparing coefficients of q in (4.7) gives the result. \square

The authors of [5] verified that $p^3 \nmid C(p)$ for every prime $p \equiv 2 \pmod{3}$ less than 32,500. Here we prove

Lemma 4.4. *For every odd prime $p \equiv 2 \pmod{3}$, we have $p \nmid C(p)$.*

Proof. Suppose by way of contradiction that $p \equiv 2 \pmod{3}$ is an odd prime with $p \mid C(p)$. Then (4.4) gives

$$\Theta^3(\phi_p) \equiv 0 \pmod{p},$$

which implies that for some coefficients A_p we have

$$\phi_p \equiv q^{-p} + \sum_{n \geq 1} A_p(np) q^{np} \pmod{p}.$$

Since ϕ_2 has no zeros on the upper half plane (and does not vanish at any cusp), we have $h_p := \phi_p \phi_2^{-p} \in M_{2p-2}(9)$. Moreover,

$$h_p \equiv \sum_{n \geq p} D_p(pn) q^{pn} \equiv q^p + \cdots \pmod{p}.$$

Therefore $h_p|U(p)|V(p) \equiv h_p \pmod{p}$ so that $w_p(h_p) = pw_p(h_p|U(p))$. Since $w_p(h_p) \equiv 2p-2 \pmod{p-1}$ and $w_p(h_p) \equiv 0 \pmod{p}$, we must have $w_p(h_p) = 0$, but this is impossible since $M_0(9)$ contains no non-constant elements. \square

Proof of Theorem 4.1. Assertion (4.2) follows from Lemma 4.3, Lemma 4.4, and the fact that $C(2) = 2$. Next, we use Proposition 4.2 and (2.1) to write

$$\frac{G|U(p^{2m+1})}{C(p^{2m+1})} - g_1 = \frac{1}{C(p^{2m+1})} \left(p^{6m+3} G_{p^{2m+1}} - \sum_{j=1}^{2m+1} p^{3j} G|U(p^{2m+1-j})|V(p^j) \right). \quad (4.8)$$

Since $C(n) = 0$ for any $n \not\equiv 2 \pmod{3}$, Proposition 4.2 and (2.1) give

$$\sum_{j=1}^{2m+1} p^{3j} G|U(p^{2m+1-j})|V(p^j) = p^3 G|T_4(p^{2m})|V(p) = p^{6m+3} G_{p^{2m}}|V(p) \equiv 0 \pmod{p^{6m+3}}.$$

The result follows from (4.8) and (4.2). \square

5. AN EXAMPLE IN WEIGHT 3 AND LEVEL 16

In [2] the authors establish an analogous representation of a weight 3 cusp form as a p -adic limit. Let χ denote the non-trivial Dirichlet character modulo 4, and define

$$g_2(z) := \eta^6(4z) = \sum_{n \geq 1} a(n) q^n = q - 6q^5 + 9q^9 + \cdots \in S_3(16, \chi),$$

$$L_2(z) := \frac{\eta^6(8z)}{\eta^2(4z)\eta^4(16z)} = \frac{1}{q} + 2q^3 - q^7 - 2q^{11} + \cdots,$$

$$H(z) := g_2(z)L_2^2(z) = \sum_{n \geq -1} C(n) q^n = \frac{1}{q} - 2q^3 - 13q^7 + 26q^{11} + \cdots.$$

The two formulas stated in the main theorem of [2] involve the hypergeometric function ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; z)$; after rewriting they are equivalent to the following statement: for every prime $p \equiv 3 \pmod{4}$ with $p^2 \nmid C(p)$ we have

$$\lim_{m \rightarrow \infty} \frac{H|U(p^{2m+1})}{C(p^{2m+1})} = g_2(z).$$

Here we prove

Theorem 5.1. *For every prime $p \equiv 3 \pmod{4}$ and every integer $m \geq 0$ we have*

$$v_p(C(p^{2m+1})) = 2m, \quad (5.1)$$

$$v_p\left(\frac{H|U(p^{2m+1})}{C(p^{2m+1})} - g_2\right) \geq 2m + 2. \quad (5.2)$$

We give only a sketch of the proof.

Proposition 5.2. *We have the following.*

(1) *For every odd integer $m \geq -1$, there exists a unique $H_m \in M_3^\infty(16, \chi) \cap \mathbb{Z}[[q]]$ of the form*

$$H_m = q^{-m} + O(q^3).$$

(2) *Let p be an odd prime and let n be a nonnegative integer. Then we have*

$$H|T_{3,\chi}(p^n) = \chi(p^n)p^{2n}H_{p^n} + C(p^n)g_2.$$

Proof. For each integer $r \geq 0$ define

$$E_r(z) := g_2(z)L_2^r(z) = \frac{\eta^{6r}(8z)}{\eta^{2r-6}(4z)\eta^{4r}(16z)} \in M_3^\infty(16, \chi).$$

We construct the form H_m with the desired properties by taking an appropriate linear combination of E_r , and uniqueness follows since $S_3(16, \chi)$ is one-dimensional. Assertion (2) is proved as before. \square

Lemma 5.3. *If $p \equiv 3 \pmod{4}$ is prime and $m \geq 0$ then*

$$C(p^{2m+1}) \equiv p^{2m}C(p) \pmod{p^{2m+2}}.$$

Proof. For each $l \geq 2$, let $\phi_l \in M_{-1}^\infty(16, \chi)$ be the form given in [2, Lem. 3.3]. We have $\phi_2(z) = \frac{\eta^2(8z)}{\eta^4(16z)}$. For $l \geq 3$ we have

$$\phi_l(z) = \phi_2(z)P_l(L_2(z)),$$

where $P_l(x) \in \mathbb{Z}[x]$ has $\deg P_l = l - 2$. Let $p \equiv 3 \pmod{4}$ be prime. As above we find that

$$\phi_p(z) = q^{-p} - C(p)q + \sum_{n \geq 5} A_p(n)q^n.$$

It follows from Proposition 2.1 that

$$\Theta^2(\phi_p) = p^2q^{-p} - C(p)q + O(q^5) \in M_3^\infty(16, \chi), \quad (5.3)$$

and we deduce using Proposition 5.2 that

$$H|U(p) = H|T_{3,\chi}(p) + p^2H|V(p) = -\Theta^2(\phi_p) + p^2H|V(p).$$

Iteratively applying $U(p^2)$ results in

$$H|U(p^{2m+1}) = - \sum_{l=0}^m p^{2(m-l)} \Theta^2(\phi_p) |U(p^{2l}) + p^{2(m+1)} H|V(p),$$

so we have

$$H|U(p^{2m+1}) \equiv -p^{2m} \Theta^2(\phi_p) \pmod{p^{2m+2}}. \quad (5.4)$$

Comparing coefficients gives the result. \square

Lemma 5.4. *For every prime $p \equiv 3 \pmod{4}$ we have $p \nmid C(p)$.*

Proof. Suppose by way of contradiction that $p \mid C(p)$. Then (5.3) and Lemma 5.2 show that $\Theta^2(\phi_p) \equiv 0 \pmod{p}$, whence

$$\phi_p \equiv q^{-p} + \sum_{n \geq 1} A_p(np) q^{np} \pmod{p}.$$

Let $f(z) = \frac{\eta^{12}(16z)}{\eta^6(8z)} = q^6 + 6q^{14} + O(q^{22}) \in M_3(16, \chi)$. Then $h_p := \phi_p f^p \in M_{3p-1}(16)$ has the form

$$h_p \equiv \sum_{n \geq 5p} D_p(pn) q^{pn} \equiv q^{5p} + \cdots \pmod{p},$$

so that

$$h_p \equiv h_p|U(p)|V(p) \pmod{p}.$$

Analyzing the filtration yields $w_p(h_p) = 2p$ and $w_p(h_p|U(p)) = 2$. However, we find by examining a basis that there is no form $h_0 \in M_2^{(p)}(16)$ with $h_0 \equiv q^5 + \cdots \pmod{p}$. This provides the desired contradiction. \square

The proof of Theorem 5.1 follows as before.

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